

# GEOMETRIC PROOFS OF SOME RESULTS OF MORITA

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## 1. INTRODUCTION

In this note we consider three results of Morita: [21, (1.3)], [22, (1.7)] and [24, (5.1)], which give relations between certain two dimensional cohomology classes of various moduli spaces of curves. We have reformulated Morita's results in more geometric language to facilitate their application in our work [13] on the Arakelov geometry of  $\mathcal{M}_g$  and to advertise them to algebraic geometers. We give a new, and hopefully more geometric, proof of each result. In the last section, we give a geometric interpretation of another of Morita's results [21, (5.4)].

Denote the mapping class group of a closed orientable surface of genus  $g$  by  $\Gamma_g$  and the mapping class group of a pointed, closed orientable surface by  $\Gamma_g^1$ .

Denote the moduli space of smooth projective curves over  $\mathbb{C}$  (i.e., compact Riemann surfaces of genus  $g$ ) by  $\mathcal{M}_g$  and the universal curve over it by  $\mathcal{C}_g$ . We denote the moduli space of principally polarized abelian varieties of dimension  $g$  by  $\mathcal{A}_g$ . As is customary, we view each of these as an orbifold, or, more accurately, a stack in the sense of [26]. The orbifold fundamental groups of  $\mathcal{M}_g$ ,  $\mathcal{C}_g$  and  $\mathcal{A}_g$  are isomorphic to

$$\Gamma_g, \Gamma_g^1 \text{ and } Sp_g(\mathbb{Z})$$

respectively, (except when  $g = 1$  when  $\pi_1(\mathcal{C}_1, *) \cong SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$  and  $\Gamma_1 = \Gamma_1^1 \cong SL_2(\mathbb{Z})$ ); the isomorphisms are unique up to inner automorphisms. Each of these moduli spaces is a  $K(\pi, 1)$  in the orbifold sense, so that for each of these moduli spaces  $X$

$$H^\bullet(X, \mathbb{Q}) \cong H^\bullet(\pi_1(X), \mathbb{Q})$$

where  $\pi_1(X)$  denotes its orbifold fundamental group. The period map  $\mathcal{M}_g \rightarrow \mathcal{A}_g$  takes the moduli point  $[C]$  of the  $C$  to that of its jacobian  $[\text{Jac } C]$ . It corresponds to the canonical homomorphism  $\Gamma_g \rightarrow Sp_g(\mathbb{Z})$ .

Suppose that  $l$  is a positive integer. The moduli space of smooth projective curves of genus  $g$  with a level  $l$  structure will be denoted by  $\mathcal{M}_g[l]$  and the universal curve over it by  $\mathcal{C}_g[l]$ . Definitions of these moduli spaces and the corresponding mapping class groups,  $\Gamma_g[l]$  and  $\Gamma_g^1[l]$ , can be found in [12, §3, §7.4]. Note that for all  $l$ ,

$$H^\bullet(\mathcal{M}_g, \mathbb{Q}) = H^\bullet(\mathcal{M}_g[l], \mathbb{Q})^{Sp_g(\mathbb{Z}/l\mathbb{Z})} \text{ and } H^\bullet(\mathcal{C}_g, \mathbb{Q}) = H^\bullet(\mathcal{C}_g[l], \mathbb{Q})^{Sp_g(\mathbb{Z}/l\mathbb{Z})}.$$

In particular, the cohomology of moduli space with no level is always a subspace of the cohomology of the moduli space with a level structure.

The *Hodge bundle* is the rank  $g$  vector bundle over  $\mathcal{A}_g$  whose fiber over the moduli point  $[A]$  of the abelian variety  $A$  is  $H^{1,0}(A)$ . Its first Chern class is denoted  $\lambda$ , as is its pullback to  $\mathcal{M}_g$  along the period map. It is known that for each  $g \geq 1$ ,

*Date:* February 8, 2008.

1991 *Mathematics Subject Classification.* Primary 14H10; Secondary, 20J05.

The first author was supported in part by grants from the National Science Foundation.

$\lambda$  generates  $H^2(\mathcal{A}_g)$  and  $H^2(\mathcal{M}_g)$ . (In fact, in both cases, the determinant  $\mathcal{L}$  of the Hodge bundle generates the Picard group of the corresponding moduli functor, [2].)

Let  $\omega$  be the first Chern class of the relative *cotangent* bundle  $\omega$  of the projection  $\mathcal{C}_g \rightarrow \mathcal{M}_g$ . It is well known that  $H^2(\mathcal{C}_g, \mathbb{Q})$  has basis  $\lambda$  and  $\omega$ . The class  $\omega$  is often denoted by  $\psi$  in the physics literature.<sup>1</sup>

One has the universal abelian variety  $\mathcal{J} \rightarrow \mathcal{A}_g$ . This has (orbifold) fundamental group isomorphic to  $Sp(H_{\mathbb{Z}}) \ltimes H_{\mathbb{Z}}$ , where  $H_{\mathbb{Z}}$  denotes the first homology group of the reference abelian variety. The splitting is given by the zero section. The pullback of  $\mathcal{J}$  along the period map is the jacobian of the universal curve.

Since the universal abelian variety is a flat bundle of real tori, the projection is a foliated bundle. It follows that there is a closed 2-form on  $\mathcal{J}$  whose restriction to each fiber is the invariant form that corresponds to the polarization (i.e., the invariant bilinear form on  $H_1(A)$ ), that is parallel with respect to the flat structure, and whose restriction to the zero section is trivial (cf. [21, §1]). We shall denote the cohomology class of this form by

$$\phi \in H^2(\mathcal{J}, \mathbb{Q}) \cong H^2(Sp_g(\mathbb{Z}) \ltimes H_{\mathbb{Z}}, \mathbb{Q}).$$

The class  $\phi$  is not integral, but  $2\phi$  is, [21, (1.2)].

Denote the canonical divisor of the Riemann surface  $C$  by  $K_C$ . Define a map  $\kappa : \mathcal{C}_g \rightarrow \mathcal{J}$  from the universal curve to the universal abelian variety by taking the point  $[C, x]$  of  $\mathcal{C}_g$  to

$$(2g - 2)x - K_C \in \text{Jac } C$$

**Theorem 1** (Morita [22, (1.7)]). *For all  $g \geq 1$ ,*

$$\kappa^* \phi = 2g(g - 1)\omega - 6\lambda \in H^2(\mathcal{C}_g, \mathbb{Q}).$$

A theta characteristic of a compact Riemann surface  $C$  is a square root of its canonical bundle  $K_C$ ; that is, a divisor class  $\alpha$  such that  $2\alpha = K_C$ . Each theta characteristic  $\alpha$  determines a divisor  $\Theta_\alpha$  in the jacobian of  $C$ . If one passes to the level 2 moduli space  $\mathcal{M}_g[2]$ , one can consistently choose a theta characteristic for each curve.<sup>2</sup> In this case we can define a map

$$j_\alpha : \mathcal{C}_g[2] \rightarrow \mathcal{J}[2]$$

by taking  $[C, x]$  to  $(g - 1)x - \alpha \in \text{Jac } C$ . Observe that  $\kappa = 2j_\alpha$ .

For each such  $\alpha$ , there is a theta divisor  $\Theta_\alpha$  in  $\mathcal{J}[2]$ , the universal abelian variety over  $\mathcal{A}_g[2]$ , the moduli space of principally polarized abelian varieties with a level 2 structure. Any two such theta divisors differ by a point of order 2 of  $\mathcal{J}[2]$ . The rational homology class of  $\Theta_\alpha$  is therefore independent of  $\alpha$  and is the pullback of a class in  $H^2(\mathcal{J}[2], \mathbb{Q})$  that we denote by  $\theta$ .

The following result is presumably well known; a proof is given in Section 3.

<sup>1</sup>Recall that  $\kappa_1 = 12\lambda$ . Morita denotes  $\kappa_1$  by  $e_1$  and  $-\psi$  by  $e$ .

<sup>2</sup>Actually, one has such sections defined over the moduli space of pairs  $(C, \alpha)$  where  $C$  is a smooth projective curve of genus  $g$  and  $\alpha$  is a theta characteristic. This moduli space has two components; one for the even theta characteristics, the other for the odd ones. Since theta characteristics correspond the  $\mathbb{Z}/2$  quadratic forms on  $H_1(C, \mathbb{Z}/2)$  associated to the intersection pairing, all theta characteristics will be defined over  $\mathcal{M}_g[2]$ , the moduli space of compact Riemann surfaces with a full level 2 structure. We prefer to work with this space which dominates the two moduli spaces of curves with theta characteristic. It is known by the results of Harer [16] and Foisy [7] that the second cohomology with  $\mathbb{Q}$  coefficients of  $\mathcal{C}_g[2]$  and the spin moduli spaces of pointed curves are isomorphic to  $H^2(\mathcal{C}_g, \mathbb{Q})$ , but we do not need such deep results in this paper.

**Proposition 2.** *For all  $g \geq 1$ ,*

$$\phi = \theta - \lambda/2 \in H^2(\mathcal{J}[2], \mathbb{Q}).$$

Since  $\phi$  and  $\lambda$  both lie in the subspace  $H^2(\mathcal{J}, \mathbb{Q})$ , it follows that  $\theta \in H^2(\mathcal{J}, \mathbb{Q})$ .

We shall prove the following result which is easily seen, using the proposition, to be equivalent to Theorem 1.

**Theorem 3.** *For all  $g \geq 1$ ,*

$$j_\alpha^* \theta = \binom{g}{2} \omega - \lambda \in H^2(\mathcal{C}_g[2], \mathbb{Q}).$$

Denote the square

$$\mathcal{C}_g \times_{\mathcal{M}_g} \mathcal{C}_g$$

of the universal curve by  $\mathcal{C}_g^2$ . It is the moduli space of triples  $(C, x, y)$  where  $x$  and  $y$  are arbitrary points of  $C$ , a compact Riemann surface. There is a canonical projection  $\mathcal{C}_g^2 \rightarrow \mathcal{M}_g$ . We then have a commutative square

$$\begin{array}{ccc} \mathcal{C}_g^2 & \xrightarrow{\delta} & \mathcal{J} \\ \downarrow & & \downarrow \\ \mathcal{M}_g & \longrightarrow & \mathcal{A}_g \end{array}$$

where  $\delta$  is the difference map which is defined by

$$\delta : [C, x, y] \mapsto [x] - [y] \in \text{Jac } C.$$

The diagonal copy of  $\mathcal{C}_g$  in  $\mathcal{C}_g^2$  is a divisor and thus has a class  $\Delta$  in  $H^2(\mathcal{C}_g^2, \mathbb{Q})$ . For  $j = 1, 2$ , denote the first Chern class of the relative cotangent bundle of the  $j$ th projection  $p_j : \mathcal{C}_g^2 \rightarrow \mathcal{C}_g$  by  $\psi_j$ . (That is,  $\psi_j = p_j^*(\omega)$ .)

**Theorem 4** (Morita [21, (1.3)]). *For all  $g \geq 1$ ,*

$$\delta^*(\phi) = \Delta + (\psi_1 + \psi_2)/2 \in H^2(\mathcal{C}_g^2, \mathbb{Q}).$$

Combining this with Proposition 2 we obtain:

**Corollary 5.** *For all  $g \geq 1$ ,*

$$\delta^*(\theta) = \Delta + (\lambda + \psi_1 + \psi_2)/2 \in H^2(\mathcal{C}_g^2[2], \mathbb{Q}).$$

In order to state the third theorem, we need a generalization of the construction of  $\mathcal{J}$ , the universal abelian variety. Suppose that  $W_{\mathbb{Z}}$  is an  $Sp_g(\mathbb{Z})$  module and that the  $Sp_g(\mathbb{Z})$  action on  $W_{\mathbb{R}} := W_{\mathbb{Z}} \otimes \mathbb{R}$  extends to an action of the Lie group  $Sp_g(\mathbb{R})$ . One can then form the corresponding flat (orbifold) bundle  $\mathcal{J}(W)$  of tori over  $\mathcal{A}_g$ . There are three ways to view this:

(i)  $\mathcal{J}(W)$  is

$$(Sp_g(\mathbb{Z}) \ltimes W_{\mathbb{Z}}) \backslash (Sp_g(\mathbb{R}) \ltimes W_{\mathbb{R}}) / U(g)$$

which maps to  $\mathcal{A}_g = Sp_g(\mathbb{Z}) \backslash Sp_g(\mathbb{R}) / U(g)$ .

(ii)  $\mathcal{J}(W)$  is the quotient of the flat bundle over  $\mathcal{A}_g$  associated to  $W_{\mathbb{R}}$  by the local system over  $\mathcal{A}_g$  corresponding to  $W_{\mathbb{Z}}$ .

(iii) If  $W_{\mathbb{Z}}$  underlies a variation of Hodge structure of odd weight over  $\mathcal{A}_g$ , then  $\mathcal{J}(W)$  can be identified with the corresponding family of Griffiths intermediate jacobians over  $\mathcal{A}_g$ .

There are three cases of interest to us:

- (i)  $W_{\mathbb{Z}} = H_{\mathbb{Z}}$  where  $\mathcal{J}(W) = \mathcal{J}$ , the universal abelian variety.
- (ii)  $W_{\mathbb{Z}} = \Lambda^3 H_{\mathbb{Z}}$ ; this is a variation of Hodge structure and  $\mathcal{J}(\Lambda^3 H)$  is the bundle of intermediate jacobians associated with the local system over  $\mathcal{A}_g$  with fiber  $H_3(A, \mathbb{Z})$  over  $[A]$ .
- (iii)  $W_{\mathbb{Z}} = \Lambda^3 H_{\mathbb{Z}}/H_{\mathbb{Z}}$  where  $H_{\mathbb{Z}}$  is imbedded in  $\Lambda^3 H_{\mathbb{Z}}$  by wedging with the polarization  $\zeta \in \Lambda^2 H_{\mathbb{Z}}$ . This is a variation of Hodge structure and  $\mathcal{J}(\Lambda^3 H/H)$  is the bundle of intermediate jacobians associated with the primitive part of the local system over  $\mathcal{A}_g$  whose fiber over  $[A]$  is the ‘primitive part’ of  $H_3(A)$ .

Suppose now that  $W_{\mathbb{Z}}$  has an  $Sp_g(\mathbb{Z})$ -invariant skew symmetric bilinear form  $q$  (as each of the examples above does.) This corresponds to an  $Sp_g(\mathbb{Z})$  invariant cohomology class  $q \in H^2(W_{\mathbb{R}}/W_{\mathbb{Z}}, \mathbb{Z})$ . The projection  $\mathcal{J}(W) \rightarrow \mathcal{A}_g$  is a foliated bundle of tori. Consequently, there is a closed 2-form on  $\mathcal{J}(W)$  whose restriction to each fiber is the invariant form that represents the class of  $q$ , that is parallel with respect to the flat structure, and whose restriction to the zero section is trivial (cf. [21, §1]). We shall denote the cohomology class of this form in  $H^2(\mathcal{J}(W), \mathbb{Q})$  by  $\phi(W)$ . The class  $2\phi(W)$  is always integral.

There are lifts  $\nu : \mathcal{M}_g \rightarrow \mathcal{J}(\Lambda^3 H/H)$  and  $\mu : \mathcal{C}_g \rightarrow \mathcal{J}(\Lambda^3 H)$  of the period maps

$$\mathcal{M}_g \rightarrow \mathcal{A}_g \text{ and } \mathcal{C}_g \rightarrow \mathcal{A}_g$$

such that the diagram

$$\begin{array}{ccc} \mathcal{C}_g & \xrightarrow{\mu} & \mathcal{J}(\Lambda^3 H) \\ \downarrow & & \downarrow \\ \mathcal{M}_g & \xrightarrow{\nu} & \mathcal{J}(\Lambda^3 H/H) \end{array}$$

commutes and the composite of  $\mu$  with the map  $\mathcal{J}(\Lambda^3 H) \rightarrow \mathcal{J}(H)$  induced by the contraction  $c : \Lambda^3 H \rightarrow H$  defined in Section 5 is the map  $\kappa : \mathcal{C}_g \rightarrow \mathcal{J}(H)$  of Theorem 1. The map  $\mu$  is the normal function of the algebraic cycle in the universal jacobian  $\mathcal{J} \rightarrow \mathcal{M}_g$  whose fiber over the point  $[C, x]$  of  $\mathcal{C}_g$  is the algebraic cycle  $C_x - C_x^-$  in  $\text{Jac } C$ . More details can be found in [8, §6], [12] and [27]. The period maps  $\nu$  and  $\mu$  induce homomorphisms

$$\Gamma_g \rightarrow Sp_g(\mathbb{Z}) \ltimes (\Lambda^3 H_{\mathbb{Z}}/H_{\mathbb{Z}}) \text{ and } \Gamma_g^1 \rightarrow Sp_g(\mathbb{Z}) \ltimes \Lambda^3 H_{\mathbb{Z}}$$

on fundamental groups. In both cases, the restriction of this homomorphism to the Torelli group is twice the Johnson homomorphism ([8, (6.3)], see also [9]).<sup>3</sup>

Each of  $H$ ,  $\Lambda^3 H$  and  $\Lambda^3 H/H$  has an  $Sp_g(\mathbb{Z})$  invariant skew symmetric bilinear form:

- (i) The form on  $H$  is the intersection pairing,  $(x, y)$ .
- (ii) The form on  $L := \Lambda^3 H$  is defined by

$$\langle x_1 \wedge x_2 \wedge x_3, y_1 \wedge y_2 \wedge y_3 \rangle = \det(x_i, y_j).$$

- (iii) The form on  $L/H = \Lambda^3 H/H$  is constructed in Section 5.

Each of these forms is primitive in the sense that it cannot be divided by an integer and still be integer valued.

Set

$$\phi_H = \kappa^* \phi(H), \phi_L = \mu^* \phi(L) \text{ and } \phi_{L/H} = \nu^* \phi(L/H).$$

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<sup>3</sup>The homomorphism induced by  $\mu$  was first constructed by Morita in [23]. The equality of Morita’s homomorphism and the one induced by the normal function follows from results in [9].

Note that  $\phi(H)$  equals the class  $\phi$  defined previously and that  $\phi_{L/H} = 0$  when  $g = 1, 2$ .

The third result of Morita we wish to prove is:

**Theorem 6** (Morita [24, (5.1)]). <sup>4</sup> For all  $g \geq 1$ ,

$$\omega = \frac{1}{2g(2g+1)}(2\phi_H + 3\phi_L) \in H^2(\mathcal{C}_g, \mathbb{Q}).$$

By some easy computations in linear algebra (cf. Corollary 19) we have

$$(g-1)\phi_L = \phi_H + \phi_{L/H}.$$

Combining this with Theorem 1, one arrives at the following equivalent result which we shall prove in Section 6:

**Theorem 7** (Morita [24, (5.8)]). For all  $g \geq 1$ ,

$$2\phi_{L/H} = (8g+4)\lambda \in H^2(\mathcal{M}_g, \mathbb{Z}).$$

*Remark 8.* (i) Since  $H_1(\Gamma_g^n, \mathbb{Z})$  is torsion free when  $g \geq 3$ , there is no torsion in  $H^2(\Gamma_g^n, \mathbb{Z})$  when  $g \geq 3$ . So if a relation between integral cohomology classes holds in  $H^2(\Gamma_g^n, \mathbb{Q})$ , it holds in  $H^2(\Gamma_g^n, \mathbb{Z})$ .

(ii) The  $(8g+4)\lambda$  appears frequently in identities involving divisor classes on  $\mathcal{M}_g$ , such as in the work of Cornalba and Harris [4] and Moriwaki [25]. These papers are related to Theorem 7 as will be explained in a forthcoming paper by the first author.

(iii) Theorem 7 also gives a clean direct proof of the non-triviality of the central extension in [8].

*Acknowledgements:* We would like to thank the referee for making useful suggestions that lead to the simplification and clarification of the proof of Theorem 3. In particular, the statement of Theorem 12 was suggested by the referee.

## 2. WEIERSTRASS POINTS

Throughout this section,  $C$  denotes a smooth projective curve of genus  $g$ , where  $g > 1$ . A point  $P$  of  $C$  is a *Weierstrass point* if there is a non-constant rational function  $f : C \rightarrow \mathbb{P}^1$  of degree  $g$  such that  $f^{-1}(\infty) = P$ . Equivalently,  $P$  is a Weierstrass point if and only if  $h^0(gP) > 1$ . The multiplicity of  $P$  as a Weierstrass point is  $h^0(gP) - 1$ . The Weierstrass point divisor of  $C$  is

$$W_C = \sum_{P \in C} (h^0(\mathcal{O}(gP)) - 1)P.$$

It has degree  $g(g-1)(g+1)$ .

We shall denote the group of divisor classes of degree  $d$  on  $C$  by  $\text{Pic}^d C$  and the  $d$ th symmetric power of  $C$  by  $C^{(d)}$ . The divisor class map gives a canonical morphism  $\mu_d : C^{(d)} \rightarrow \text{Pic}^d C$  whose image is the locus of effective divisor classes of degree  $d$ , which we shall denote by  $W_d$ . The map of  $C^{(d)}$  to  $W_d$  is well known to be of degree 1 when  $1 \leq d \leq g$ . It is classical that  $\mu_1$  is an imbedding. We shall identify  $W_1$  with  $C$  via  $\mu_1$  and denote  $W_{g-1}$  by  $\Theta_C$ .

<sup>4</sup>One should note that Morita's class  $z_1$  equals  $2\phi_H$  and his class  $z_2$  equals  $3\phi_L$ . We will prove this in Section 7.

Define

$$\sigma_C : C \times C \rightarrow \text{Pic}^{g-1} C$$

by  $\sigma_C(P, Q) = gP - Q$ .

**Lemma 9.** *There is a divisor  $E_C$  in  $C$ , with the same support as  $W_C$ , such that*

$$\sigma_C^* \Theta_C = g\Delta + p_1^* E_C,$$

where  $p_1 : C \times C \rightarrow C$  is projection onto the first factor.

*Proof.* First, suppose that  $P$  and  $Q$  are two distinct points of  $C$ . If  $gP - Q \in \Theta_C$ , then there exist  $P_1, \dots, P_{g-1} \in C$  such that

$$gP \equiv Q + P_1 + \dots + P_{g-1}.$$

Since  $P \neq Q$ , this implies that  $h^0(\mathcal{O}(gP)) > 1$  and that  $P$  is a Weierstrass point. It follows that

$$\sigma_C^* \Theta_C = n\Delta + p_1^* E_C$$

where  $n \in \mathbb{Z}$  and  $E_C$  is a divisor in  $C$  with the same support at  $W_C$ .

To determine  $n$ , pick a point  $P$  of  $C$  that is not a Weierstrass point. Then, by the computation above, the image of  $C$  in  $\text{Pic}^{g-1} C$  under the mapping

$$\nu_P : Q \mapsto gP - Q$$

intersects  $\Theta_C$  only at  $(g-1)P$ . The Poincaré dual of  $\Theta_C$  is the polarization

$$a_1 \wedge b_1 + \dots + a_g \wedge b_g \in H^2(\text{Pic}^{g-1} C, \mathbb{Z})$$

where  $a_1, \dots, b_g$  is a symplectic basis of  $H^1(C, \mathbb{Z})$ . Integrating this over the image of  $C$ , we see that the degree of the pullback of  $\Theta_C$  to  $C$  along  $\nu_P$  is  $g$ . It follows that

$$\nu_P^* \Theta_C = gP$$

and that  $n = g$ . □

Define

$$j_C : C \rightarrow \text{Pic}^{g-1} C$$

by  $j_C(P) = (g-1)P$ . Note that  $j$  is just the restriction of  $\sigma_C$  to the diagonal in  $C \times C$ . Since the normal bundle of the diagonal in  $C \times C$  is the tangent bundle of  $C$ , we have:

**Corollary 10.** *There is an isomorphism of line bundles*

$$j_C^* \mathcal{O}(\Theta_C) \cong \omega_C^{\otimes (-g)} \otimes \mathcal{O}(E_C). \quad \square$$

**Lemma 11.** *We have  $\deg E_C = \deg W_C$ .*

*Proof.* The degree of  $j_C^* \Theta_C$  on  $C$  is easily seen to be  $g(g-1)^2$ . By Lemma 11, the degree of  $E_C$  equals the degree of  $\mathcal{O}(\Theta_C) \otimes \omega_C^{\otimes g}$ , which is

$$2g(g-1) + g(g-1)^2 = g(g-1)(g+1) = \deg W_C.$$

□

We can now apply this to the universal curve  $\mathcal{C}_g$  over  $\mathcal{M}_g$ . We have the mapping

$$\sigma : \mathcal{C}_g^2 \rightarrow \text{Pic}_{\mathcal{M}_g}^{g-1} \mathcal{C}_g$$

whose restriction to the fiber over  $[C] \in \mathcal{M}_g$  is  $\sigma_C$ . Denote the Weierstrass point divisor in  $\mathcal{C}_g$  by  $W$  and the projection onto the first factor by  $p_1 : \mathcal{C}_g^2 \rightarrow \mathcal{C}_g$ . The universal theta divisor in  $\text{Pic}_{\mathcal{M}_g}^{g-1} \mathcal{C}_g$  will be denoted by  $\Theta$ . It has fiber  $\Theta_C$  over  $[C] \in \mathcal{M}_g$ .

**Theorem 12.** *On  $\mathcal{C}_g^2$ , we have  $\sigma^* \Theta = g\Delta + p_1^* W$ .*

*Proof.* Set  $E = \sigma^* \Theta - g\Delta$ . By Lemma 9, the support of  $E$  equals that of  $W$ . By Lemma 11,  $E$  and  $W$  have the same degree over  $\mathcal{M}_g$ . Since the Weierstrass points of the generic curve are distinct,  $E$  and  $W$  must be equal.  $\square$

Restricting  $\sigma$  to the diagonal copy of  $\mathcal{C}_g$  in  $\mathcal{C}_g^2$ , we obtain a morphism

$$j : \mathcal{C}_g \rightarrow \text{Pic}_{\mathcal{M}_g}^{g-1} \mathcal{C}_g$$

whose restriction to the fiber over  $[C] \in \mathcal{M}_g$  is  $j_C$ .

**Theorem 13.** *As line bundles over  $\mathcal{C}_g$ ,*

$$j^* \mathcal{O}(\Theta) \cong \omega^{\otimes(-g)} \otimes \mathcal{O}(W).$$

This follows from the previous result and the following lemma.

**Lemma 14.** *The normal bundle of  $\mathcal{C}_g$  imbedded diagonally in  $\mathcal{C}_g^2$  is the relative tangent bundle  $\omega^{-1}$  of  $\mathcal{C}_g \rightarrow \mathcal{M}_g$ .*

*Proof.* Let  $\pi : \mathcal{C}_g \rightarrow \mathcal{M}_g$  denote the projection. The two projections of  $\mathcal{C}_g^2$  induce a map  $T\mathcal{C}_g^2 \rightarrow T\mathcal{C}_g \oplus T\mathcal{C}_g$ . We then have an exact sequence

$$0 \longrightarrow T\mathcal{C}_g \xrightarrow{\Delta} (T\mathcal{C}_g \oplus T\mathcal{C}_g)|_{\text{diag}} \xrightarrow{f} T\mathcal{C}_g \longrightarrow T\mathcal{M}_g \longrightarrow 0$$

where  $\Delta$  is the diagonal map and  $f(v, w) = \pi_*(v) - \pi_*(w)$ . It follows that the normal bundle is the kernel of  $T\mathcal{C}_g \rightarrow T\mathcal{M}_g$ , which is the relative tangent bundle.  $\square$

### 3. PROOF OF THEOREM 3

We will prove Theorem 3 by comparing the formula for the class of Weierstrass point locus  $W$  derived in the previous section with a second expression for the class of  $W$  derived using Wronskians. This second expression is due to Arakelov [1, Lemma 3.3] as was pointed out to us by Jean-Benoit Bost.

**Proposition 15** (Arakelov). *Over  $\mathcal{C}_g$  we have*

$$c_1(\mathcal{O}(W)) = \binom{g+1}{2} w - \lambda.$$

*Proof.* Suppose that  $\zeta_1(t), \dots, \zeta_g(t)$  is a local holomorphic framing of the Hodge bundle  $\pi_* \Omega_{\mathcal{C}_g/\mathcal{M}_g}^1$  over  $\mathcal{M}_g$ . Then

$$W(\zeta_1, \dots, \zeta_g) \otimes (\zeta_1 \wedge \dots \wedge \zeta_g)^{-1}$$

is a section of  $\omega^{\otimes \binom{g+1}{2}} \otimes \mathcal{L}^{-1}$  that is independent of the choice of the framing, and therefore extends to a global section over  $\mathcal{C}_g$ . Here

$$W(\zeta_1, \dots, \zeta_g) := \begin{vmatrix} f_1(t, z) & \dots & f_g(t, z) \\ \vdots & \ddots & \vdots \\ \frac{\partial^{g-1}}{\partial z^{g-1}} f_1(t, z) & \dots & \frac{\partial^{g-1}}{\partial z^{g-1}} f_g(t, z) \end{vmatrix} dz^{\binom{g+1}{2}}$$

is the Wronskian, where  $\zeta_j(t) = f_j(t, z)dz$  locally. It is a section of  $\omega^{\otimes \binom{g+1}{2}}$ . The result follows as  $W$  is the divisor of this section by classical Riemann surface theory.  $\square$

*Proof of Theorem 3.* Choose a theta characteristic  $\alpha$  defined over  $\mathcal{C}_g[2]$ . This gives an isomorphism

$$\text{Pic}_{\mathcal{M}_g}^{g-1} \mathcal{C}_g \xrightarrow{\alpha} \text{Jac}_{/\mathcal{M}_g} \mathcal{C}_g$$

The composition of  $j$  with this mapping is the mapping

$$j_\alpha : \mathcal{C}_g \rightarrow \mathcal{J} = \text{Jac}_{/\mathcal{M}_g} \mathcal{C}_g$$

in the introduction. Denote the image of  $\Theta$  in the universal jacobian by  $\Theta_\alpha$ . It follows from Theorem 13 that

$$[W] = j_\alpha^* \theta + gw \in H^2(\mathcal{C}_g[2], \mathbb{Q}).$$

On the other hand, Arakelov's computation implies that

$$[W] = \binom{g+1}{2} w - \lambda.$$

Equating these two expressions, we see that

$$j_\alpha^* \theta = \binom{g}{2} w - \lambda.$$

$\square$

*Proof of Proposition 2.* First note that

$$H^\bullet(\mathcal{J}[2], \mathbb{Q}) \cong H^\bullet(Sp_g(\mathbb{Z})[2] \ltimes H_{\mathbb{Z}}, \mathbb{Q})$$

By a theorem of Raghunathan [28],  $H^1(Sp_g[2], H_{\mathbb{Q}})$  vanishes and, by a theorem of Borel [3],  $H^2(Sp_g(\mathbb{Z})[2], \mathbb{Q})$  has rank one when  $g \geq 2$ . Since  $\phi$  is not zero in  $H^2(\mathcal{J}[2], \mathbb{Q})$ , it follows, by looking at the Hochschild-Serre spectral sequence of the group extension

$$0 \rightarrow H_{\mathbb{Z}} \rightarrow Sp_g(\mathbb{Z})[2] \ltimes H_{\mathbb{Z}} \rightarrow Sp_g(\mathbb{Z})[2] \rightarrow 1,$$

that  $H^2(\mathcal{J}[2], \mathbb{Q})$  is two dimensional and spanned by  $\lambda$  and  $\phi$ . By restricting the class  $\theta$  to any fiber of  $\mathcal{J}[2]$ , we see that

$$\theta = \phi + c\lambda.$$

To determine the rational number  $c$ , we restrict  $\theta$  to the zero section of  $\mathcal{J}[2]$ . If we choose  $\alpha$  to be even (i.e.,  $\vartheta_\alpha$  is an even theta function), then we see that the restriction of  $\theta$  to the zero section is the divisor corresponding to the theta null corresponding to  $\vartheta_\alpha$ . Since the theta nulls are sections of a square root of the determinant  $\mathcal{L}$  of the Hodge bundle, as follows from the transformation law for theta nulls, we see that  $c = 1/2$ . The proof in the case when  $g = 1$  is simpler and left to the reader.  $\square$



## 4. PROOF OF THEOREM 4

This result is not nearly as deep as the previous result. It follows from standard computations when  $g \geq 3$  that  $H^2(\mathcal{C}_g, \mathbb{Q})$  is four dimensional and has basis  $\lambda$ ,  $\psi_1$ ,  $\psi_2$  and  $\Delta$ , the dual of the diagonally imbedded copy of  $\mathcal{C}_g$ . It follows that

$$\delta^*(\phi) = a\lambda + b\psi_1 + c\psi_2 + d\Delta$$

where the coefficients are rational numbers. It follows from Proposition 14 that the restriction of  $\Delta$  to the diagonal is  $-\psi$ . So the restriction of  $\delta^*(\phi)$  to the diagonal is

$$a\lambda + (b + c - d)\psi \in H^2(\mathcal{C}_g, \mathbb{Q}).$$

The image of the restriction of  $\delta : \mathcal{C}_g^2 \rightarrow \mathcal{J}$  to the diagonal  $\mathcal{C}_g$  has image contained in the zero section of  $\mathcal{J}$ . Since the restriction of  $\phi$  to the zero section of  $\mathcal{J}$  is trivial, it follows that  $a\lambda + (b + c - d)\psi = 0$  in  $H^2(\mathcal{C}_g, \mathbb{Q})$ . So  $a = 0$  and  $b + c - d = 0$ . The constants  $b$ ,  $c$  and  $d$  can now be determined by restricting to any fiber  $C \times C$ . In this case, it is an elementary exercise in algebraic topology to show that under the difference map,  $\phi$  pulls back to  $\Delta + (\psi_1 + \psi_2)/2$ . The result follows.

## 5. SOME LINEAR ALGEBRA

As in the introduction, we denote the fundamental representation of  $Sp_g(\mathbb{Q})$  by  $H$  and the standard symplectic form on  $H$  by  $q_H$ . Denote its third exterior power  $\Lambda^3 H$  by  $L$  and the natural symplectic form on it, which was defined in the introduction, by  $q_L$ .

If  $a_1, \dots, a_g, b_1, \dots, b_g$  is a symplectic basis of  $H$ , then the element  $\zeta = \sum_{j=1}^g a_j \wedge b_j$  is an  $Sp_g(\mathbb{Q})$  invariant element of  $\Lambda^2 H$  which is independent of the choice of the basis. Wedging with  $\zeta$  induces an  $Sp_g(\mathbb{Q})$  invariant mapping  $H \rightarrow L$ . One also has the contraction map  $c : L \rightarrow H$  which is defined by

$$c : x \wedge y \wedge z \mapsto q_H(x, y)z + q_H(y, z)x + q_H(z, x)y.$$

**Proposition 16.** *When  $g \geq 2$ , the composite  $H \rightarrow L \rightarrow H$  is  $g - 1$  times the identity.*  $\square$

**Corollary 17.** *When  $g = 2$ ,  $c : L \rightarrow H$  is an isomorphism. Consequently,  $L/H$  is zero when  $g = 1$  and 2.*  $\square$

Denote the image in  $L/H$  of  $x \wedge y \wedge z \in L$  by  $\overline{x \wedge y \wedge z}$ . The projection  $p : L \rightarrow L/H$  has a canonical  $Sp_g(\mathbb{Q})$  equivariant splitting  $j$  after tensoring with  $\mathbb{Q}$ . It is defined by

$$j(\overline{x \wedge y \wedge z}) = x \wedge y \wedge z - \zeta \wedge c(x \wedge y \wedge z)/(g - 1).$$

An  $Sp_g(\mathbb{Q})$  invariant symplectic form on  $L/H$  is given by

$$q_{L/H}(u, v) = (g - 1)q_L(j(u), j(v)).$$

One can easily check that this form is integral and primitive. The following result follows from a straightforward computation.

**Proposition 18.** *For all  $g \geq 2$  we have  $(g - 1)q_L = p^*q_{L/H} + c^*q_H$ .*  $\square$

**Corollary 19.** *The classes  $\phi_L$ ,  $\phi_H$ ,  $\phi_{L/H}$  in  $H^2(\mathcal{C}_g, \mathbb{Q})$ , defined in the introduction, satisfy*

$$(g - 1)\phi_L = \phi_{L/H} + \phi_H \quad \square$$

## 6. PROOF OF THEOREM 7

We begin by dispensing with the cases  $g = 1$  and  $2$ . In these cases  $(8g+4)\lambda = 0$  as  $H^2(\Gamma_g, \mathbb{Z})$  is cyclic of order 12 when  $g = 1$ , and 10 when  $g = 2$ . On the other hand, as observed in Section 5,  $L/H$  is trivial in these cases too. So both  $(8g+4)\lambda$  and  $\phi_{L/H}$  vanish and the result is trivially true.

Now suppose that  $g \geq 3$ . We shall denote the mapping class group of a compact oriented surface  $S$  of genus  $g$  with  $r$  boundary components and  $n$  distinct marked points (not lying on the boundary of  $S$ ) by  $\Gamma_{g,r}^n$ . As usual,  $r$  and  $n$  are omitted when they are zero. Here we consider only  $\Gamma_g$ ,  $\Gamma_g^1$  and  $\Gamma_{g,1}$ . The natural homomorphisms

$$\Gamma_{g,1} \rightarrow \Gamma_g^1 \rightarrow \Gamma_g$$

induce homomorphisms

$$H^2(\Gamma_g, \mathbb{Z}) \hookrightarrow H^2(\Gamma_g^1, \mathbb{Z}) \twoheadrightarrow H^2(\Gamma_{g,1}, \mathbb{Z}).$$

Harer [15] has proved that the left and right hand groups have rank one and are generated by  $\lambda$  for all  $g \geq 3$ . The middle group has rank two and is generated by  $\lambda$  and  $\psi$  for all  $g \geq 3$  as can be seen by applying the Gysin sequence to the group extension

$$0 \rightarrow \mathbb{Z} \rightarrow \Gamma_{g,1} \rightarrow \Gamma_g^1 \rightarrow 1.$$

Since these groups are torsion free when  $g \geq 3$  (cf. Remark 8), and since  $2\phi_{L/H}$  is integral, it suffices to show that  $\phi_{L/H} = (4g+2)\lambda$  in  $H^2(\Gamma_g^1, \mathbb{Q})$ .

We know that

$$\phi_{L/H} = x\lambda \in H^2(\Gamma_g, \mathbb{Q})$$

for some  $x \in \mathbb{Q}$ . To determine  $x$ , we work in  $H^2(\Gamma_g^1, \mathbb{Q})$ . By Theorem 1,

$$\phi_H = 2g(1-g)\psi - 6\lambda \in H^2(\Gamma_g^1, \mathbb{Q}),$$

so that  $6\phi_{L/H} - x\phi_H$  is a multiple of  $\psi$ .

To compute the constant  $x$  we convert the problem into linear algebra. Denote the Lie algebra of the pronipotent radical of the completion of  $\Gamma_{g,r}^n$  with respect to the standard representation  $\Gamma_{g,r}^n \rightarrow Sp_g(\mathbb{Q})$  by  $\mathfrak{u}_{g,r}^n$ . (See [10] for definitions.) There is a natural homomorphism

$$H^2(\mathfrak{u}_{g,r}^n)^{Sp_g} \rightarrow H^2(\Gamma_{g,r}^n, \mathbb{Q})$$

which is an isomorphism when  $g \geq 3$ .<sup>5</sup>

Since the kernel of  $H^2(\Gamma_g^1, \mathbb{Q}) \rightarrow H^2(\Gamma_{g,1}, \mathbb{Q})$  is spanned by  $\psi$ , it follows that

$$\mathbb{Q}\psi = \ker\{H^2(\mathfrak{u}_g^1)^{Sp_g} \rightarrow H^2(\mathfrak{u}_{g,1})^{Sp_g}\}.$$

To determine the combinations of  $\psi_H$  and  $\psi_{L/H}$  that lie in the kernel of this map we use the short exact sequence

$$0 \longrightarrow \mathrm{Gr}_{-2}^W H_2(\mathfrak{u}_{g,1}) \xrightarrow{\mathrm{cup}^*} \Lambda^2 H_1(\mathfrak{u}_{g,1}) \xrightarrow{\beta} \mathrm{Gr}_{-2}^W(\mathfrak{u}_{g,1}) \longrightarrow 0.$$

The first map is the dual of the cup product and  $\beta$  is induced by the bracket. Taking invariants, we obtain the exact sequence

$$0 \longrightarrow \mathrm{Gr}_{-2}^W H_2(\mathfrak{u}_{g,1})^{Sp_g} \xrightarrow{\mathrm{cup}^*} \Lambda^2 H_1(\mathfrak{u}_{g,1})^{Sp_g} \xrightarrow{\beta} \mathrm{Gr}_{-2}^W(\mathfrak{u}_{g,1})^{Sp_g} \longrightarrow 0$$

<sup>5</sup>One can see this by first reducing to the case where  $r = n = 0$  using the exact sequence [10, (3.6)], the result [10, (8.2)], and a spectral sequence argument. The case  $r = n = 0$  can then be proved using [15] for the computation of  $H^2(\Gamma_g, \mathbb{Q})$ , [10, (7.1)] for the injectivity of the homomorphism, and [10, (11.1)] to see that  $H^2(\mathfrak{u}_g)^{Sp_g}$  is one dimensional.

Denote the fundamental representation of  $Sp_g(\mathbb{Q})$  by  $H$ . It follows from Johnson's Theorem [20] that the Johnson homomorphism induces an  $Sp_g(\mathbb{Q})$  equivariant isomorphism

$$H_1(\mathfrak{u}_{g,1}) \cong \Lambda^3 H.$$

Let  $a_1, \dots, a_g, b_1, \dots, b_g$  be a symplectic basis of  $H$  and  $A_1, \dots, A_d, B_1, \dots, B_d$  be a symplectic basis of the kernel of the contraction  $c : \Lambda^3 H \rightarrow H$  with respect to the symplectic form  $q_L$  on  $\Lambda^3 H$ , where

$$2d = \dim \Lambda^3 H / H = \binom{2g}{3} - 2g = \frac{2g(2g+1)(g-2)}{3}.$$

Note that this is where the  $8g+4 = 4(2g+1)$  in the statement of the theorem will come from. Set

$$\zeta = \sum_{j=1}^g a_j \wedge b_j \in (\Lambda^2 H)^{Sp_g}$$

and

$$f_H = \sum_{j=1}^g a_j \wedge \zeta \bigwedge b_j \wedge \zeta \text{ and } f_{L/H} = \sum_{j=1}^d A_j \bigwedge B_j$$

which are elements of  $(\Lambda^2 \Lambda^3 H)^{Sp_g}$  and are viewed as elements of  $\Lambda^2 H_1(\mathfrak{u}_{g,1})^{Sp_g}$  via the isomorphism above.

Denote the Malcev Lie algebra of the fundamental group of a surface with one boundary component by  $\mathfrak{p}_{g,1}$ . (It is isomorphic to the free pronilpotent Lie algebra generated by  $H$ .) View  $\zeta$  as the element  $\sum_j [a_j, b_j]$  of  $\text{Gr}_{-2}^W \mathfrak{p}_{g,1}$ . The canonical map

$$\alpha : \text{Gr}_{-2}^W \mathfrak{u}_{g,1} \rightarrow \text{Gr}_{-2}^W \text{Der } \mathfrak{p}_{g,1}$$

is injective.

It follows from a direct computation using [10, §11] or the fact that the Euler class of a surface of genus  $g$  is  $2 - 2g$  that

$$\alpha \circ \beta(f_H) = (2 - 2g) \text{ad}(\zeta),$$

and from [10, (11.4)] that

$$\alpha \circ \beta(f_{L/H}) = -6 \sum_{j=1}^d \frac{q_{L/H}(A_j, B_j)}{g(2g+1)} \text{ad}(\zeta) = \frac{-6d}{g(2g+1)} \text{ad}(\zeta) = -4(g-2) \text{ad}(\zeta)$$

Consequently

$$\alpha \circ \beta(2(g-2)f_H - (g-1)f_{L/H}) = 0 \text{ in } \text{Gr}_{-2}^W \text{Der } \mathfrak{p}_{g,1}$$

and therefore  $2(g-2)f_H - (g-1)f_{L/H}$  spans the one dimensional vector space  $H_2(\mathfrak{u}_{g,1})^{Sp_g}$ . Consequently

$$(1) \quad \langle 6\phi_{L/H} + x\phi_H, 2(g-2)f_H - (g-1)f_{L/H} \rangle = 0.$$

Now  $\phi_L \in H^2(\mathfrak{u}_g^1)^{Sp_g}$  takes the value

$$(c^* q_H)(f_H) = (g-1)^2 q_H(\zeta) = g(g-1)^2$$

on  $f_H$ , while  $\phi_{L/H} \in H^2(\mathfrak{u}_g^1)^{Sp_g}$  takes the value

$$(p^* q_{L/H})(f_{L/H}) = (g-1)q_L(f_{L/H}) = d(g-1)$$

on  $f_{L/H}$ . Since  $\phi_H(f_{L/H}) = \phi_{L/H}(f_H) = 0$  (by representation theory, for example), (1) becomes

$$g(g-2)(g-1)^2x - 6d(g-1)^2 = 0$$

from which it follows that  $x = 4g + 2$ .

*Remark 20.* We give a brief explanation of why Morita's classes  $z_1$  and  $z_2$  defined in [24, p. 173] are related to ours by

$$z_1 = 2\phi_H \text{ and } z_2 = 3\phi_L.$$

The first point is that his forms  $C_1$  and  $C_1$  are related to ours by

$$C_1 = 4c^*q(H) \text{ and } C_2 = 6q(L).$$

This can be seen by direct computation.

The second point is that Morita uses these quadratic forms to construct extensions of  $\frac{1}{2}\Lambda^3H$  by  $\mathbb{Z}$ , whereas we are considering extensions of  $\Lambda^3H$  by  $\mathbb{Z}$ . The isomorphism

$$(\text{multiplication by } 2) : \frac{1}{2}\Lambda^3H \rightarrow \Lambda^3H$$

multiplies quadratic forms by 4. So if we rewrite Morita's proof using the group

$$Sp_g(\mathbb{Z}) \ltimes \Lambda^3H \text{ instead of } Sp_g(\mathbb{Z}) \ltimes \frac{1}{2}\Lambda^3H$$

we need to replace  $C_1$  by  $C_1/4$  and  $C_2$  by  $C_2/4$ .

The final point is that Morita's construction of an extension of  $\Lambda^3H$  by  $\mathbb{Z}$  from a skew symmetric form  $q$  (given at the top of page 174 of [24]) yields one whose Chern class is twice the cohomology class in  $H^2(\Lambda^3H, \mathbb{Z})$  corresponding to  $q$ . This may be seen by computing commutators in the corresponding Heisenberg groups.

Putting all this together, we have  $z_1 = 2(4\phi_H/4) = 2\phi_H$  and  $z_2 = 2(6\phi_L/4) = 3\phi_L$ .

## 7. REMARKS ON [21, (5.4)]

This result of Morita states that if  $g \geq 2$ , then

$$H_1(\Gamma_g, H_{\mathbb{Z}}) \cong \mathbb{Z}/(2g-2)\mathbb{Z}.$$

Since  $H_{\mathbb{Z}}$  has an  $Sp_g(\mathbb{Z})$  invariant unimodular form, it is isomorphic to its dual as a  $\Gamma_g$  module. So, by the universal coefficient theorem,

$$\text{torsion subgroup of } H^2(\Gamma_g, H_{\mathbb{Z}}) \cong \mathbb{Z}/(2g-2)\mathbb{Z}.$$

**Proposition 21.** *For all  $g \geq 1$ ,  $H^2(\Gamma_g, H_{\mathbb{Q}})$  vanishes.*

*Proof.* Harer [15] has shown that the natural homomorphism  $\Gamma_{g,1} \rightarrow \Gamma_g$  induces an isomorphism on  $H^2$  with  $\mathbb{Q}$  coefficients for all  $g \geq 2$ . One can now use the Gysin sequence

$$0 \rightarrow H^0(\Gamma_{g,1}, \mathbb{Q}) \rightarrow H^2(\Gamma_g^1, \mathbb{Q}) \rightarrow H^2(\Gamma_{g,1}, \mathbb{Q}) \rightarrow H^1(\Gamma_g^1, \mathbb{Q}) \dots$$

and the fact that the second Betti number of  $\Gamma_g$  is 0 in genus 2 and 1 when  $g \geq 3$  [15] to show that

$$\dim H^2(\Gamma_g^1, \mathbb{Q}) = 1 + \dim H^2(\Gamma_g, \mathbb{Q}) = \begin{cases} 1 & g = 2; \\ 2 & g \geq 3. \end{cases}$$

Now consider the Hochschild-Serre spectral sequence of the group extension

$$1 \rightarrow \pi_1(\text{reference surface}, *) \rightarrow \Gamma_g^1 \rightarrow \Gamma_g \rightarrow 1.$$

By passing to subgroups of level  $\geq 4$  and applying Deligne's degeneration result [5], we see that this spectral sequence (for cohomology with  $\mathbb{Q}$  coefficients) degenerates at the  $E_2$  term. This implies that

$$\dim H^2(\Gamma_g^1, \mathbb{Q}) = 1 + \dim H^2(\Gamma_g, \mathbb{Q}) + \dim H^1(\Gamma_g, H_{\mathbb{Q}}).$$

It follows that  $H^1(\Gamma_g, H_{\mathbb{Q}})$  vanishes for all  $g \geq 2$ . The result is easily proved when  $g = 1$  using the “center kills trick” —  $-I$ , being central in  $\Gamma_1 = SL_2(\mathbb{Z})$  acts trivially on  $H^\bullet(\Gamma_1, H_{\mathbb{Q}})$  on the one hand, and as  $-I$  on the other as it acts this way in  $H$ , which forces  $H^\bullet(SL_2(\mathbb{Z}), H_{\mathbb{Q}})$  to be trivial.  $\square$

Combining this with Morita's computation, we obtain:

**Corollary 22.** *For all  $g \geq 2$ , we have  $H^2(\Gamma_g, H_{\mathbb{Z}}) \cong \mathbb{Z}/(2g-2)\mathbb{Z}$ .*  $\square$

This result has a natural geometric interpretation:  $H^2(G, A)$ , where  $G$  is a group and  $A$  a  $G$ -module, classifies extensions of  $G$  by  $A$ ; the identity being the split extension  $G \ltimes A$ . So Morita's result says that when  $g \geq 2$ , the set of equivalence classes of central extensions of  $\Gamma_g$  by  $H_{\mathbb{Z}}$  is a cyclic group of order  $2g-2$ . It is easy to realize these extensions geometrically.

For each  $d \in \mathbb{Z}$  one has the degree  $d$  part  $\text{Pic}_{\mathcal{M}_g}^d \mathcal{C}_g \rightarrow \mathcal{M}_g$  of the relative Picard group of the universal curve. Since the base and fiber of this map are Eilenberg-MacLane spaces  $K(\pi, 1)$  (in the orbifold sense), the total space is too, and we have a short exact sequence of fundamental groups

$$0 \rightarrow H_{\mathbb{Z}} \rightarrow \pi_1(\text{Pic}_{\mathcal{M}_g}^d \mathcal{C}_g, *) \rightarrow \Gamma_g \rightarrow 1$$

as, for each curve  $C$ , there is a canonical isomorphism  $H_1(\text{Pic}_{\mathcal{M}_g}^d \mathcal{C}_g) \cong H_1(C)$ . Taking  $d$  to the class of this extension in  $H^2(\Gamma_g, H_{\mathbb{Z}})$ , one obtains a function

$$\epsilon : \mathbb{Z} \rightarrow H^2(\Gamma_g, H_{\mathbb{Z}}).$$

Using the addition map

$$\text{Pic}_{\mathcal{M}_g}^d \mathcal{C}_g \times_{\mathcal{M}_g} \text{Pic}_{\mathcal{M}_g}^e \mathcal{C}_g \rightarrow \text{Pic}_{\mathcal{M}_g}^{d+e} \mathcal{C}_g,$$

it is easy to show this function is a group homomorphism. The canonical bundle gives a section of  $\text{Pic}_{\mathcal{M}_g}^{2g-2} \mathcal{C}_g$ . Consequently, this homomorphism factors through  $\mathbb{Z}/(2g-2)\mathbb{Z}$ .

**Proposition 23.** *If  $g \geq 3$ ,  $\epsilon(d) = 0$  if and only if  $d$  is divisible by  $2g-2$ .*

*Proof.* We have already proved that if  $d$  is divisible by  $2g-2$ , then  $\epsilon(d) = 0$ . Suppose that  $0 < d \leq 2g-2$  generates  $\ker \epsilon$ . Then  $d|(2g-2)$ . Set  $e = (2g-2)/d$ . Since  $\epsilon(d) = 0$ , there is a smooth section  $s$  of  $\text{Pic}_{\mathcal{M}_g}^d \mathcal{C}_g \rightarrow \mathcal{M}_g$ . Then  $e \cdot s - (\text{canonical bundle})$  is a smooth section of the universal jacobian and therefore determines a cohomology class in  $H^1(\Gamma_g, H_{\mathbb{Z}})$ . It follows from Johnson's theorems that this group is torsion (see, for example, [9, (5.2)]). By the universal coefficient theorem,  $H^1(\Gamma_g, H_{\mathbb{Z}}) \cong H_0(\Gamma_g, H_{\mathbb{Z}})$ , which is trivial. It follows that the class of the section  $e \cdot s - (\text{canonical bundle})$  is trivial, and therefore that  $s$  is homotopic to a section of  $\text{Pic}_{\mathcal{M}_g}^d \mathcal{C}_g$  corresponding to an  $e$ th root of the canonical bundle. But by the solution to the Francetta Conjecture [2] (see also [9, §12]), this implies that  $d$  is divisible by  $2g-2$ , and therefore that  $d = 2g-2$ .  $\square$

Combining this with Morita's Theorem, we obtain:

**Theorem 24.** *If  $g \geq 3$ , then the map  $\mathbb{Z}/(2g-2)\mathbb{Z} \rightarrow H^2(\Gamma_g, H_{\mathbb{Z}})$  that takes  $d$  to the class corresponding to  $\text{Pic}_{\mathcal{M}_g}^d \mathcal{C}_g$  is an isomorphism.*  $\square$

Similarly, one can use the solution of the Francetta Conjecture for moduli spaces of curves with a level [9, §12] to prove that if  $g \geq 3$  the image of the restriction map

$$H^2(\Gamma_g, H_{\mathbb{Z}}) \rightarrow H^2(\Gamma_g[l], H_{\mathbb{Z}})$$

is  $\mathbb{Z}/(2g-2)\mathbb{Z}$  if  $l$  is odd and  $\mathbb{Z}/(g-1)\mathbb{Z}$  if  $l$  is even and positive, where  $\Gamma_g[l]$  denotes the level  $l$  subgroup of  $\Gamma_g$ . We do not know how to prove surjectivity when  $l > 0$ .

#### REFERENCES

- [1] S. Arakelov: *Families of algebraic curves with fixed degeneracies*, Izv. Acad. Nauk. SSSR, Ser. Mat. Tom 35 (1971), in Russian. Translation in: Math. USSR Izvestija, vol. 5 (1971), 1277–1302.
- [2] E. Arbarello, M. Cornalba: *The Picard group of the moduli space of curves*, Topology 26 (1987), 153–171.
- [3] A. Borel: *Stable real cohomology of arithmetic groups*, Ann. Sci. Ecole Norm. Sup. 7 (1974), 235–272.
- [4] M. Cornalba, J. Harris: *Divisor classes associated to families of stable varieties, with applications to the moduli space of curves*, Ann. Scient. Éc. Sup., t. 21 (1988), 455–475.
- [5] P. Deligne: *Théorème de Lefschetz et critères de dégénérescence de suites spectrales*, Inst. Hautes Études Sci. Publ. Math. 35 (1968), 259–278.
- [6] D. Eisenbud, J. Harris: *The monodromy of Weierstrass points*, Invent. Math. 90 (1987), 333–341.
- [7] J. Foisy: *The second homology group of the level 2 mapping class group of an orientable surface*, preprint, 1997.
- [8] R. Hain: *Completions of mapping class groups and the cycle  $C - C^-$* , in *Mapping Class Groups and Moduli Spaces of Riemann Surfaces*, C.-F. Bödigheimer and R. Hain, editors, Contemp. Math. 150 (1993), 75–105.
- [9] R. Hain: *Torelli groups and Geometry of Moduli Spaces of Curves*, in *Current Topics in Complex Algebraic Geometry* (C. H. Clemens and J. Kollar, eds.) MSRI publications no. 28, Cambridge University Press, 1995, 97–143.
- [10] R. Hain: *Infinitesimal presentations of the Torelli groups*, J. Amer. Math. Soc. 10 (1997), 597–651.
- [11] R. Hain: *Moriwaki's inequality and generalizations*, in preparation.
- [12] R. Hain, E. Looijenga: *Mapping class groups and moduli spaces of curves*, in *Algebraic Geometry—Santa Cruz 1995*, Proc. Sympos. Pure Math., 62, Part 2 (1997), 97–142.
- [13] R. Hain, D. Reed: *On the Arakelov geometry of moduli spaces of curves*, in preparation.
- [14] J. Harer: *The second homology group of the mapping class group of an orientable surface*, Invent. Math. 72 (1983), 221–239.
- [15] J. Harer: *The third homology group of the moduli space of curves*, Duke Math. J. 63 (1991), 25–55.
- [16] J. Harer: *The rational Picard group of the moduli spaces of Riemann surfaces with spin structure*, Contemp. Math. 150 (1993), 107–136.
- [17] B. Harris: *Harmonic volumes*, Acta Math. 150 (1983), 91–123.
- [18] D. Johnson: *An abelian quotient of the mapping class group  $\mathcal{I}_g$* , Math. Ann. 249 (1980), 225–242.
- [19] D. Johnson: *The structure of the Torelli group—II: A characterization of the group generated by twists on bounding curves*, Topology 24 (1985), 113–126.
- [20] D. Johnson: *The structure of the Torelli group—III: The abelianization of  $\mathcal{I}$* , Topology 24 (1985), 127–144.
- [21] S. Morita: *Families of jacobian manifolds and characteristic classes of surface bundles, I*, Ann. Inst. Fourier, Grenoble 39 (1989), 777–810.
- [22] S. Morita: *Families of jacobian manifolds and characteristic classes of flat bundles, II*, Math. Proc. Camb. Phil. Soc., 105 (1989), 79–101.

- [23] S. Morita: *The extension of Johnson's homomorphism from the Torelli group to the mapping class group*, Invent. Math. 111 (1993), 197–224.
- [24] S. Morita: *A linear representation of the mapping class group of orientable surfaces and characteristic classes of surface bundles*, in the Proceedings of the 37th Taniguchi Symposium on Topology and Teichmüller Spaces, July 1995, S. Kojima et al editors, World Scientific (1996), 159–186.
- [25] A. Moriwaki: *A sharp slope inequality for general stable fibrations of curves*, J. Reine Angew. Math. 480 (1996), 177–195.
- [26] D. Mumford: *Towards an enumerative geometry of the moduli space of curves*, in Arithmetic and Geometry, M. Artin and J. Tate, editors (1983), Birkhäuser, 271–328.
- [27] M. Pulte: *The fundamental group of a Riemann surface: mixed Hodge structures and algebraic cycles*, Duke Math. J. 57 (1988), 721–760.
- [28] M. Raghunathan: *Cohomology of arithmetic subgroups of algebraic groups: I*, Ann. of Math. (2) 86 (1967), 409–424.

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